Chapter 3

Valid inequalities

Consider the general integer linear programming problem from Chapter 1 and Chapter 2.

$$\begin{aligned} z_{IP} = \max\ & c^T x \\ \text{s. t. } & Ax \le b \\ & x \in \mathbb{Z}^n_+ \end{aligned} \tag{IP}$$

Denote the set of feasible solutions of (IP) as

$$X = \{x \in \mathbb{Z}^n_+ \mid Ax \le b\}.$$

We saw in Chapter 2 that the convex hull of X is a polyhedron, meaning that it can be written (*in theory*) as

$$conv(X) = \{x \ge 0 \mid \tilde{A}x \le \tilde{b}\}.$$

If we knew $\tilde{A}$ and $\tilde{b}$, then we could solve (IP) as a linear program and efficiently obtain an optimal solution. However, we do not know the exact description $\tilde{A}$ and $\tilde{b}$ of $conv(X)$ in general.

In Chapter 2 we saw that X can be described using different formulations, and that some formulations are better than others. A related idea is that, once we are given a formulation P of X, we can get "closer" to $conv(X)$ using *valid inequalities*. Before formally defining valid inequalities, Figure 3.1 gives a geometrical illustration of the idea. If we add a new constraint (in green) to our original formulation (in blue), such that we cut off part of P *without cutting off any integer point* of X, we can potentially get a better approximation of $conv(X)$. Most commercial and non-commercial solvers use branch-and-cut, an algorithm based on branch-and-bound where at each node, instead of directly solving the formulation given by the LP relaxation of a model, many inequalities (such as the green one in Figure 3.1) are added. In this way, one can obtain a better formulation, potentially leading to better bounds, before branching.

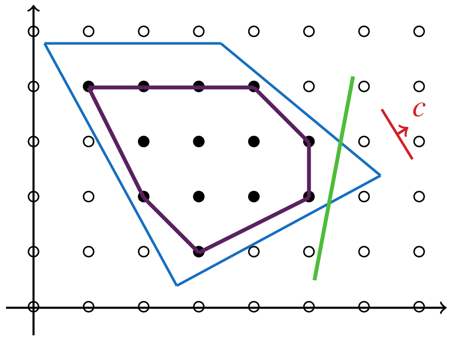


Figure 3.1: Geometrical illustration of a valid inequality for $conv(X)$.

Definition 16. *An inequality $\pi^T x \leq \pi_0$ is called* valid inequality *for a set $X \subseteq \mathbb{R}^n$ if $\pi^T \bar{x} \leq \pi_0$ for all $\bar{x} \in X$.*

Note that in the definition above, X can be any set, either discrete or continuous. The following observation will be useful for our purposes.

Observation 5. *If an inequality is valid for an integer set of points $X \in \mathbb{Z}^n$, then it is also valid for $conv(X)$.*

Obviously, if $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then inequalities $Ax \leq b$ are valid for P. The question we would like to answer in this chapter is how to find further valid inequalities. We first start with some precise examples and then we define how valid inequalities can be derived in general. Then, we give algorithms to automate the process of generating valid inequalities, and finally we introduce some ideas on how to find "good" valid inequalities. (This chapter uses many examples and follows a similar structure to Chapters 8 and 9 of [3].)

3.1 First examples

Example 14 (Stable Set). *Consider the stable set problem* (2.24)-(2.26) *for a graph $G = (V, E)$ with $n = |V|$. As a reminder, the* stable set polytope *is the convex hull of feasible solutions of* (2.24)-(2.26)*.*

$$STAB(G) = conv\{x \in \{0,1\}^n \mid x_i + x_j \leq 1, \text{ for all } (i,j) \in E\}.$$

The LP-relaxation of model (2.24)-(2.26)*, called the* edge relaxation*, with set of feasible solutions*

$$ESTAB(G) = \{x \in [0,1]^n \mid x_i + x_j \leq 1, \text{ for all } (i,j) \in E\}.$$

In Section 2.3 we provided a different formulation for the stable set problem, the clique relaxation*, which was defined as*

$$QSTAB(G) = \{x \in [0,1]^{|V|} \mid \sum_{i \in Q} x_i \leq 1, \ \ \forall Q \text{ clique of } G\}. \tag{3.1}$$

Actually, the clique inequalities

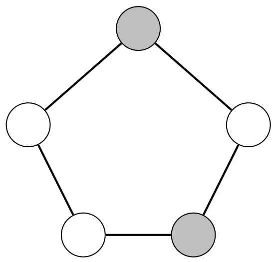
$$\sum_{i \in Q} x_i \leq 1, \quad \forall Q \text{ clique of } G \tag{3.2}$$

are valid inequalities for $STAB(G)$. We can check Definition 16 for the integer points in $STAB(G)$. Assume for a contradiction that $\sum_{i \in Q} x_i \leq 1$ is not satisfied for a certain clique Q. Then, there exist at least to vertices $i, j \in Q$ with $x_i = x_j = 1$, but this would contradict inequality $x_i + x_j \leq 1$, because since Q is a clique, there exists an edge $(i, j) \in E$. By Observation 5, we know that the clique inequalities are valid for $STAB(G)$.

Another class of valid inequalities for $STAB(G)$ are the odd cycle inequalities. Consider a cycle $C \subseteq V$ of odd length, $|C| = 2k + 1$, where $k \geq 1$ is an integer. The odd cycle inequality *for C is defined as*

$$\sum_{i \in C} x_i \leq \frac{|C| - 1}{2}. \tag{3.3}$$

The odd cycle inequalities are valid for the integer points in $STAB(G)$ because for every odd cycle C we can at most select $\frac{|C|-1}{2}$ vertices to be in the stable set.



Example 15 (Implications). *Consider the set of binary points*

$$X = \{x \in \{0, 1\}^5 \mid 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leq -2\}$$

Clearly, if $x_2 = x_4 = 0$, then the inequality defining X cannot be satisfied. This means that the inequality $x_2 + x_4 \geq 1$ is valid for X. We call this type of inequality implication inequality.

Example 16 (Knapsack). *Consider the Knapsack set*

$$X = \{x \in \{0, 1\}^5 \mid 3x_1 + 4x_2 + 2x_3 + 3x_4 + x_5 \leq 5\}$$

Clearly, x_1 and x_2 cannot be set to 1 at the same time, otherwise the inequality defining X would not be satisfied. This means that the inequality $x_1 + x_2 \leq 1$ is valid for X.

In general, consider now a Knapsack constraint for n items, each of size a_i, $i = 1, \dots, n$ and capacity b

$$\sum_{i=1}^{n} a_i x_i \leq b \tag{3.4}$$

The Knapsack polytope *is the convex hull of all binary points satisfing* (3.4)

$$conv\{x \in \{0,1\}^n \mid \sum_{i=1}^{n} a_i x_i \leq b\}.$$

A set $C \subseteq \{1, \dots, n\}$ is called a cover *if and only if $\sum_{i \in C} a_i > b$. If, in addition, $\sum_{i \in C \setminus \{j\}} a_i \leq b$, for all $j \in C$, then C is called a* minimal cover. *For any cover C the* cover inequality

$$\sum_{i \in C} x_i \leq |C| - 1$$

is a valid inequality for the Knapsack polytope.

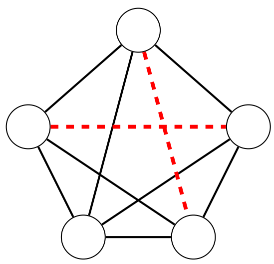
Example 17 (Matching). *Consider the maximum cardinality matching problem for a graph $G = (V, E)$, introduced in Example 2.*

$$\begin{aligned} \max \quad & \sum_{(i,j) \in E} x_{ij} & & (1.2)\\ s.\ t. \quad & \sum_{(i,j) \in \delta(i)} x_{ij} \leq 1 & \forall i \in V & \\ & x_{ij} \in \{0,1\} & \forall (i,j) \in E & \end{aligned}$$

Consider a subset of vertices $T \subseteq V$ of odd cardinality, $|T| = 2k + 1$, where $k \geq 1$ is an integer. Let $E(T) = \{(i,j) \in E \mid i, j \in T\}$ be the set of edges with both endpoints in T. Then, the odd set inequality

$$\sum_{(i,j) \in E(T)} x_{ij} \leq \frac{|T| - 1}{2} \tag{3.5}$$

is valid for the convex hull of feasible solutions of (1.2)*. Indeed, since $|T|$ is of odd cardinality, it cannot be the case that all edges selected to be in the matching have both endpoints in T.*



Example 18 (Integer Rounding). *In this example we present a simple "trick" that will be generalized later. Consider the integer set $X = P \cap \mathbb{Z}^4$ where*

$$P = \{x \in \mathbb{R}^4_+ \mid 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\}.$$

Multiplying the inequality defining P by $u_1 = \frac{1}{11}$ we obtain an inequality that is clearly valid for P.

$$\frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq \frac{72}{11}$$

Moreover, since $x_1, \ldots, x_4 \geq 0$*, we can round up the coefficients of the variables, and obtain*

$$2x_1 + 2x_2 + x_3 + x_4 = \left\lceil \frac{13}{11} \right\rceil x_1 + \left\lceil \frac{20}{11} \right\rceil x_2 + \lceil 1 \rceil x_3 + \left\lceil \frac{6}{11} \right\rceil x_4 \geq \frac{72}{11}$$

Finally, for $x = (x_1, x_2, x_3, x_4) \in X$*, since both the variables and the coefficients are integer numbers, and* $\frac{72}{11} = 6.5454...$*, inequality*

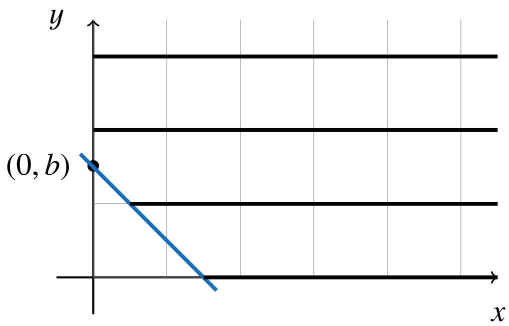
$$2x_1 + 2x_2 + x_3 + x_4 \geq \left\lceil \frac{72}{11} \right\rceil = 7$$

holds, meaning that it is valid for X*.*

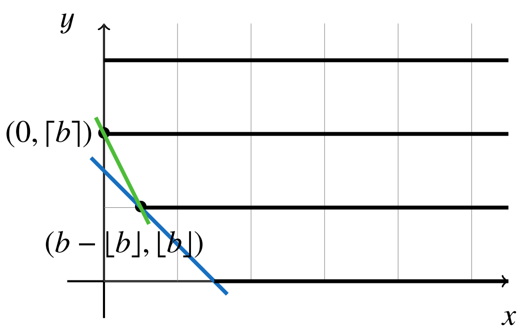
Example 19 (Mixed-integer Rounding). *Consider a mixed-integer set*

$$X^{\geq} = \{(x, y) \in \mathbb{R}^1_{\geq 0} \times \mathbb{Z}^1 \mid x + y \geq b, x \geq 0\}$$

where $b \notin \mathbb{Z}$*. Below we illustrate* $\cancel{X^{\geq 0}}$ $X^{\geq}$ *when* $b = 1.5$*.*



Geometrically, it is easy to see that the green inequality below is a valid inequality which provides a better formulation than only $x + y \geq b$*. In addition, this inequality is easy to calculate, one only has to compute the equation of the line going through points* $(0, \lceil b \rceil)$ *and* $(b - \lfloor b \rfloor, \lfloor b \rfloor)$*. (Point* $(0, \lceil b \rceil)$ *is the intersection between* $x = 0$ *and* $y = \lceil b \rceil$*, point* $(b - \lfloor b \rfloor, \lfloor b \rfloor)$ *is the intersection between* $x + y = b$ *and* $y = \lfloor b \rfloor$*.)*



In this way, one obtains that inequality

$$x + fy \geq f\lceil b \rceil$$

with $f = b - \lfloor b \rfloor$, *is valid for* $X^{\geq}$.

Moreover, the inequality above can be extended to a mixed integer set with more than two variables. Consider for example $\cancel{X^{\geq 0}}\ X^{\geq} = P \cap (\mathbb{Z}^4 \times \mathbb{R}^1)$ *where*

$$P = \{(y, s) \in \mathbb{R}^4_{\geq 0} \times \mathbb{R}^1_{\geq 0} \mid 13y_1 + 20y_2 + 11y_3 + 6y_4 + s \geq 72\}$$

Divide the constraint defining P *by* 11

$$\frac{13}{11}y_1 + \frac{20}{11}y_2 + y_3 + \frac{6}{11}y_4 + \frac{1}{11}s \geq \frac{72}{11}.$$

and round up the coefficients of the y *variables. We obtain an inequality which is valid for* P

$$2y_1 + 2y_2 + y_3 + y_4 + \frac{1}{11}s \geq \frac{72}{11}.$$

Set $y = 2y_1 + 2y_2 + y_3 + y_4$ *and* $x = \frac{1}{11}s$. *We are in the setting above, with* $x + y \geq \frac{72}{11}$. *In this case,* $f = \frac{72}{11} - \left\lfloor \frac{72}{11} \right\rfloor = \frac{72}{11} - 6 = \frac{72}{11} - \frac{66}{11} = \frac{6}{11}$.

We know that inequality

$$x + \frac{6}{11}y \geq \frac{6}{11} \cdot 7$$

is valid for $\cancel{X^{\geq 0}}\ X^{\geq}$, *hence*

$$\frac{11}{6}x + y \geq 7$$

is also valid, which in terms of the original variables, produces

$$2y_1 + 2y_2 + y_3 + y_4 + \frac{1}{6}s \geq 7.$$

3.2 Valid inequalities in general

In many cases, valid inequalities are constructed by using knowledge about a specific problem, such odd cycles inequalitie for the stable set problem or odd set inequalities for the matching problem. However, there also exist general procedures to generate valid inequalities, which can be very useful when problem-specific knowledge is not available.

3.2.1 Valid inequalities for Linear Programs

Although we use valid inequalities to better solve integer programs, we are actually interested in generating valid inequalities for polyhedra (for formulations). More precisely, given a set of integer points X, and a (not ideal) formulation P for it, we want inequalities that are valid for $conv(X)$ while cutting off some points in P.

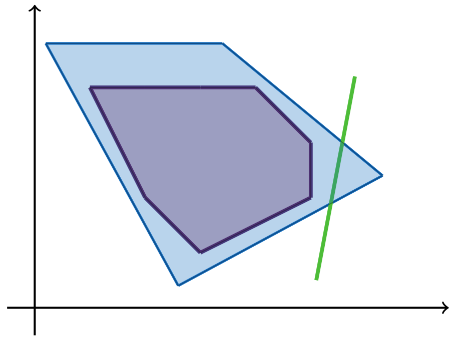


Figure 3.2: Valid inequalities for *conv*(X) cutting off part of the blue formulation.

It is therefore useful to better understand the geometrical meaning of valid inequalities in the context of linear programs.

Theorem 8. *Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. $\pi^T x \leq \pi_0$ is valid for $P = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\} \neq \emptyset$ if, and only if, there exist $u \in \mathbb{R}^m_{\geq 0}$ such that $u^T A \geq \pi^T$ and $ub \leq \pi_0$.*

Proof. By linear programming duality, $\max\{\pi^T x \mid x \in P\} \leq \pi_0$ if, and only if, $\min\{u^T b \mid u^T A \geq \pi^T, u \geq 0\} \leq \pi_0$. □

Let us look at the interpretation in the primal. Equation $\max\{\pi^T x \mid x \in P\} \leq \pi_0$ is telling us is that an inequality $\pi^T x \leq \pi_0$ is valid for P if, when optimizing over P in the direction of $\pi^T x$ we obtain a maximum which is at most π_0. Indeed, if the maximum was larger than π_0, then the inequality would not be valid for some points in P.

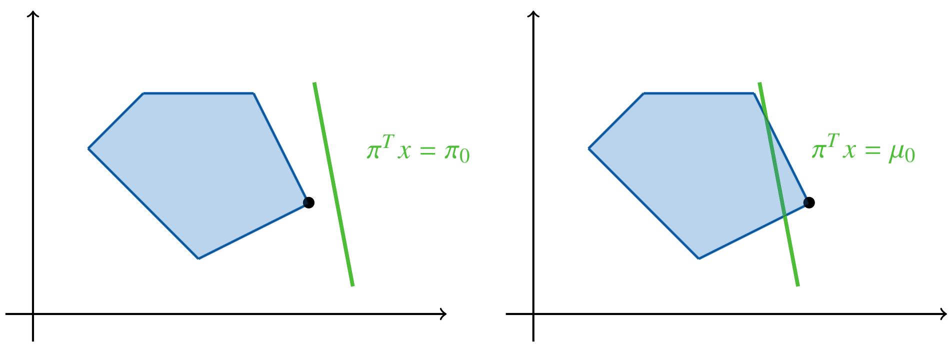


Figure 3.3: On the left, the optimal value over P in direction $\pi^T x$ is $\leq \pi_0$, so $\pi^T x \leq \pi_0$ is valid for P. On the right, the optimal value over P in direction $\pi^T x$ is $> \mu_0$, so $\pi^T x \leq \mu_0$ is not valid for P.

3.2.2 Valid inequalities for general Integer Programs

In this section we describe a general procedure to construct valid inequalities for integer programs. Let us start with a similar trick as in Example 18.

Example 20. *Let $X = P \cap \mathbb{Z}^n$ where*

$$P = \{(x_1, x_2) \in \mathbb{R}^2 \mid 7x_1 - 2x_2 \leq 14 \\ x_2 \leq 3 \\ 2x_1 - 2x_2 \leq 3 \\ x_1, x_2 \geq 0\}$$

We proceed in three steps.

Step 1. *Combine the constraints (except the nonnegativity constraints) with nonnegative weights $u = (\frac{2}{7}, \frac{37}{63}, 0)$. We obtain inequality*

$$2x_1 + \frac{1}{63}x_2 \leq \frac{121}{21},$$

which is valid for P (this inequality is implied by inequalities defining P).

Step 2. *Since $x_1, x_2 \geq 0$, we have that $2x_1 + 0x_2 = \lfloor 2 \rfloor x_1 + \left\lfloor \frac{1}{63} \right\rfloor x_2 \leq 2x_1 + \frac{1}{63}x_2$. Hence,*

$$2x_1 + 0x_2 \leq \frac{121}{21} = 5.7619...$$

is valid for P.

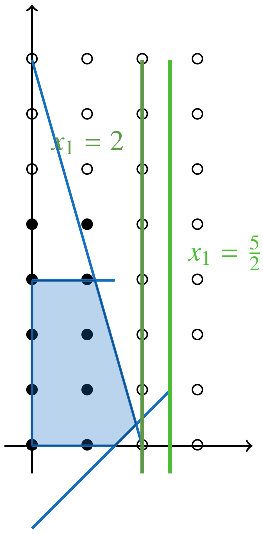
Step 3. *Finally, since points in X are integer,*

$$2x_1 \leq \left\lfloor \frac{121}{21} \right\rfloor = 5$$

is valid for points in X.

Note that the first two steps construct valid inequalities for the polyhedron P, while the last step derives a valid inequality for the set of integer points X. It is this last step which provides an inequality that potentially improves the formulation P for X.

This process can be iterated using inequality $2x_1 \leq 5$. Indeed, using a weight of $\frac{1}{2}$ for it we obtain inequality $x_1 \leq \left\lfloor \frac{5}{2} \right\rfloor = 2$. The figure below illustrates polytope P in blue, and the valid inequalities in green. Note that the inequality obtained in the second iteration is tigther than the first one.



The procedure above can be generalized to any formulation P for an integer set X using the Chvátal-Gomory procedure. Inequalities obtained with this procedure are called *Chvátal-Gomory (CG) inequalities*.

Chvátal-Gomory procedure to construct valid inequalities

Consider an integer set $X = P \cap \mathbb{Z}^n$, where $P = \{x \in \mathbb{R}^n_{\geq 0} \mid Ax \leq b\}$, and A is an $m \times n$ matrix with columns $A_1, \ldots, A_n$. Let $u \in \mathbb{R}^m_{\geq 0}$. Then,

(i) The inequality

$$\sum_{j=1}^{n} (u^T A_j) x_j \leq u^T b$$

is valid for P, because $u \geq 0$ and $\sum_{j=1}^{n} A_j x_j \leq b$.

(ii) The inequality

$$\sum_{j=1}^{n} \lfloor u^T A_j \rfloor x_j \leq u^T b$$

is valid for P, because $x \geq 0$.

(iii) The inequality

$$\sum_{j=1}^{n} \lfloor u^T A_j \rfloor x_j \leq \lfloor u^T b \rfloor$$

is valid for X, because x is integer so $\sum_{j=1}^{n} \lfloor u^T A_j \rfloor x_j$ is integer.

Note that the big difficulty of the CG procedure lies in the selection of the multipliers $u \in \mathbb{R}^m_{\geq 0}$, and therefore also on selecting the constraints that one includes in the linear combination. Some known tips to select u are have been summarized for example in [9]. First, it is a known result that if we select all multipliers to be rational and $0 \leq u_i < 1$, we do not lose generality. Second, any CG inequality can be generated using a vector u with at most $\min\{m, n\}$ strictly positive coefficients. These tips imply that the coefficients of the CG inequality will be reasonably small (if the original coefficients are reasonably small).

Note also that when P is defined by $Ax \geq b$, then the procedure slightly changes. (Which changes do you have to make in this case?)

Example 21 (Example 17 continued). *Consider formulation P corresponding to model (1.2) of the matching problem*

$$P = \{x \in [0,1]^{|E|} \mid \sum_{(i,j)\in\delta(i)} x_{ij} \leq 1, \ \forall i \in V\},$$

We will see that the odd set inequalities can be obtained by applying the Chvátal-Gomory procedure. Consider a subset of vertices $T \subseteq V$ of odd cardinality $|T| \geq 3$.

$$\sum_{(i,j)\in E(T)} x_{ij} \leq \frac{|T|-1}{2} \tag{3.5}$$

Step 1. *Define $u \in \mathbb{R}_+^{|V|}$ as follows*

$$u_i = \begin{cases} \frac{1}{2} & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$

Take a nonnegative linear combination of the constraints defining P using multipliers u, providing the valid inequality for P

$$\sum_{(i,j)\in E(T)} x_{ij} + \frac{1}{2} \sum_{(i,j)\in\delta(T)} x_{ij} \leq \frac{|T|}{2}$$

where $E(T) = \{(i,j) \in E \mid i,j \in T\}$ is the set of edges with both endpoints in T and $\delta(T) = \{(i,j) \in E \mid i \in T, j \notin T\}$ is the set of edges with exactly one endpoint in T.
Step 2. *Round down the coefficients of the left-hand-side, resulting in*

$$\sum_{(i,j)\in E(T)} x_{ij} \leq \frac{|T|}{2}$$

which is a valid inequality for P.
Step 3. *Since all variables x_{ij} are integer (when viewed as variables in $X = P \cap \mathbb{Z}^{|E|}$), $\sum_{(i,j)\in E(T)} x_{ij}$ must also be integer, hence*

$$\sum_{(i,j)\in E(T)} x_{ij} \leq \left\lfloor \frac{|T|}{2} \right\rfloor = \frac{|T|-1}{2}$$

is a valid inequality for $X = P \cap \mathbb{Z}^{|E|}$.

Note that $\left\lfloor \frac{|T|}{2} \right\rfloor = \frac{|T|-1}{2}$ because $|T|$ is of odd cardinality. For sets T of even cardinality, the procedure has no effect, because it would provide a trivial inequality.

The interesting (and rather surprising fact) is that every valid inequality can be obtained by iteratively applying the Chvátal-Gomory procedure a *finite* number of times. This is even more interesting when considering that there are infinitely many valid inequalities as illustrated in Figure 3.4

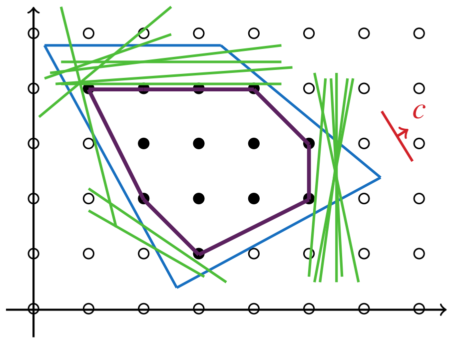


Figure 3.4: There are infinitely many possible valid inequalities (only a few here...).

Theorem 9. *Every valid inequality for $X = P \cap \mathbb{Z}^n$, where $P = \{x \in \mathbb{R}^n_{\geq 0} \mid Ax \leq b\}$, can be obtained by applying the Chvátal-Gomory procedure a finite number of times.*

The theorem above motivates the following definition.

Definition 17. *The* Chvátal rank of a valid inequality $\pi^T x \leq \pi_0$ *for $X = P \cap \mathbb{Z}^n$, where $P = \{x \in \mathbb{R}^n_{\geq 0} \mid Ax \leq b\}$, is the minimum number of iterations of the Chvátal-Gomory procedure required to obtain $\pi^T x \leq \pi_0$.*

Since a Chvátal-Gomory inequality $\lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor$ with $u \in \mathbb{R}^m_{\geq 0}$ is valid for $X = \{x \in \mathbb{Z}^n_+ \mid Ax \leq b\}$, by Observation 5, it is also valid for $P_I = conv(X)$. This means that $P_I \subseteq P \cap \{x \in \mathbb{R}^n_{\geq 0} \mid \lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor\}$

Definition 18. *Let $P = \{x \in \mathbb{R}^n_{\geq 0} \mid Ax \leq b\}$. The set*

$$P' = P \cap \bigcap_{u \geq 0} \{x \in \mathbb{R}^n_{\geq 0} \mid \lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor\}$$

is called the Chvátal closure *of P.*

Theorem 10. *The Chvátal closure P' of a polyhedron P is again a polyhedron.*

The theorem above is not obvious because P' is defined by infinitely many inequalities. More precisely, while the set of inequalities in the intersection is infinite, a finite number of these inequalities is sufficient to describe P' (Schrijver 1980). Since P' is a polyhedron, this means that we can again compute the Chvátal closure of P', and obtain a new polyhedron, and so on. This leads to a sequence of polyhedra

$$\begin{aligned} P^{(0)} &:= P \\ P^{(i)} &:= \left(P^{(i-1)}\right)' && \text{for } i \geq 1 \end{aligned}$$

such that

$$P = P^{(0)} \supseteq P^{(1)} \supseteq P^{(2)} \supseteq P^{(3)} \supseteq \cdots \supseteq P^{(k)} = P_I$$

Definition 19. *The* Chvátal rank of a polyhedron *P is the smallest k such that $P^{(k)} = P_I$.*

It is a known fact that the Chvátal rank of a polyhedron is always finite.

Example 22 (Example 21 continued). *Consider the macthing polytope*

$$P = \{x \in [0,1]^{|E|} \mid \sum_{(i,j)\in\delta(i)} x_{ij} \leq 1,\ \forall i \in V\},$$

Example 21 shows that the odd set inequalities

$$\sum_{e\in E(T)} x_e \leq \frac{|T|-1}{2} \quad \forall T \subseteq V \text{ of odd cardinality } |T| \geq 3 \tag{3.5}$$

have Chvátal rank 1*. Moreover, it can be shown that*

$$\begin{aligned} P_I = P' = \{x \in [0,1]^{|E|} \mid & \sum_{(i,j)\in\delta(i)} x_{ij} \leq 1,\ \forall i \in V \\ & \sum_{e\in E(T)} x_e \leq \frac{|T|-1}{2} \quad \forall T \subseteq V \text{ of odd cardinality } |T| \geq 3\} \end{aligned}$$

hence polytope P has Chvátal rank 1*.*

3.3 Automatic generation of valid inequalities

In the previous sections we have seen how to generate families of inequalities, either by using properties of the problem or by generating them with the Chvátal-Gomory procedure. However, many of the previously presented families of valid inequalities have exponential size. Consider for example the family of clique inequalities for the stable set polytope

$$\sum_{i\in Q} x_i \leq 1, \text{ for all } Q \subseteq V$$

We have exponentially many of them, because we have one inequality for each clique $Q \subseteq V$, and there are potentially exponentially many cliques in a given graph G. In this section we explain how to actually use valid inequalities in practice, even for families of exponetial size.

Moreover, depending on the objective function, we might not actually care about some of the inequalities, for example in Figure 3.5, the orange inequalities are less interesting than the green inequalities considering our objecive function. Indeed, for the integral optimum of this problem, all orange inequalities are automatically satisfied.

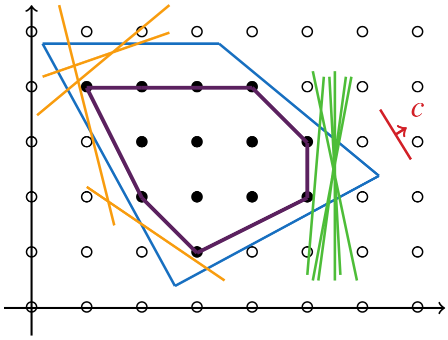


Figure 3.5: Which valid inequalities are actually interesting?

This idea of generating only "useful" inequalities inspired the idea of *automatic reformulation* or *cutting plane algorithms*. We present it here as in [3].

3.3.1 A general cutting plane algorithm

First, we need the following definition.

Definition 20. *The* separation problem *associated with a polytope $P \subseteq \mathbb{R}^n$ and a point $\bar{x} \in \mathbb{R}^n$ is to decide whether $\bar{x} \in P$ or to provide an inequality $\pi x \leq \pi_0$ satisfied by all points $x \in P$ but such that $\pi\bar{x} > \pi_0$.*

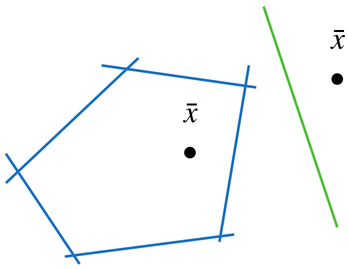


Figure 3.6: If $\bar{x}$ is not in the polytope P defined by the blue inequalities, then the green inequality separates $\bar{x}$ from P.

In our context, we use separation problems as follows. Look for example at Figure 3.5. Let $\bar{x}$ be an optimal solution obtained when optimizing the red function $c^T x$ over the blue formulation. Assume that we know a family of valid inequalities $\mathcal{F}$ (like the green ones). We would like to know whether one of the inequalities in $\mathcal{F}$ is such that $\pi x \leq \pi_0$ is satisfied for all points $x \in conv(X)$, but such that $\pi\bar{x} > \pi_0$. If such an inequality exists, then we can add it to our original (blue) formulation, and obtain a smaller formulation. As mentioned above, this is particularly interesting when the family of inequalities $\mathcal{F}$ contains exponentially many inequalities.

The following algorithm, called the cutting plane algorithm, formalizes this idea of generating only "interesting" inequalities. The input of the algorithm is a formulation P of a set $X \subseteq \mathbb{Z}^n$, and a family of valid inequalities $\mathcal{F}$.

Cutting plane algorithm

- *Initialization*. Set $t = 0$ and $P^0 = P$.
- *Iteration t*. Solve the LP
$$\bar{z}^t = \max\{cx \mid x \in P^t\}$$
and let x^t be the corresponding optimal solution.
  - If $x^t \in \mathbb{Z}^n$, then stop. x^t is an optimal integral solution.
  - If $x^t \notin \mathbb{Z}^n$, then solve the *separation problem* for x^t, $conv(X)$ and family $\mathcal{F}$.
  - If the separation problem returns an inequality from $\mathcal{F}$ such that $\pi^t x^t > \pi_0^t$, then it cuts off x^t. In this case, set $P^{t+1} = P^t \cap \{x \mid \pi^t x \leq \pi_0^t\}$ and increment t. Otherwise, stop.

If the algorithm terminates without finding an integral solution for IP, then

$$P^t = P \cap \{x \mid \pi^i x \leq \pi_0^i,\ i = 1, \ldots, t\}$$

is a better (or equal) formulation than P and can, for example, be given as an initial improved formulation for a branch-and-bound algorithm. The branch-and-cut algorithm is a branch-and-bound algorithm where a cutting plane procedure is integrated at each node.

Figure 3.7 illustrates the cutting plane algorithm.

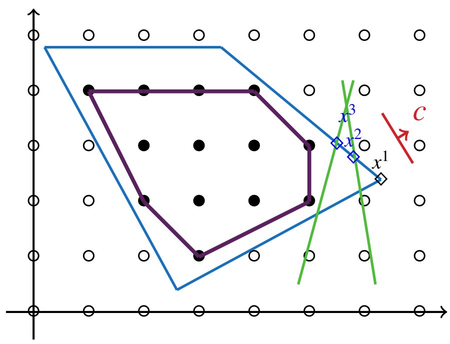


Figure 3.7: Cutting plane algorithm (points x^2 and x^3 have been updated for the illustration to be completely accurate (previously they were not optimal w.r.t c).

As a final remark, note that the cutting plane algorithm heavily relies on knowing how to solve the separation problem for a family of valid inequalities $\mathcal{F}$. This is in general not a trivial task, and the separation problem can even be NP-hard. We will discuss separation problems later in this chapter, for now we only give an example.

Example 23 (**Example 22**). *Let $G = (V, E)$ be a graph. The* maximum weight matching *is a generalization of the maximum cardinality matching problem, where we associate nonnegative weights w_{ij} to every edge $(i, j) \in E$ and we want to maximize the total weight of the matching.*

$$\max \sum_{(i,j)\in E} w_{ij}x_{ij} \tag{3.6}$$

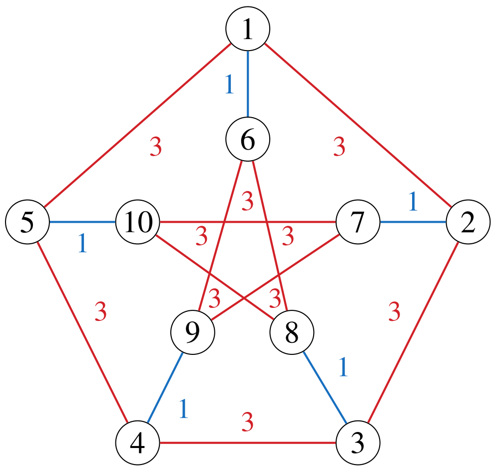
$$s.\ t. \sum_{(i,j)\in\delta(i)} x_{ij} \leq 1 \qquad i \in V \tag{3.7}$$

$$x_{ij} \in \{0, 1\} \qquad (i, j) \in E \tag{3.8}$$

(The maximum cardinality matching problem is a particular case where $w_{ij} = 1$ for all $(i, j) \in E$.)

We will illustrate the cutting plane algorithm on the following graph using the family of odd set inequalities

$$\sum_{(i,j)\in E(T)} x_{ij} \leq \frac{|T| - 1}{2} \quad \forall T \subseteq V \textit{ of odd cardinality } |T| \geq 3. \tag{3.5}$$



Let P^0 be the formulation corresponding to model (3.6)-(3.8)

$$P^0 = \{x \in [0, 1]^{|E|} \mid \sum_{(i,j)\in\delta(i)} x_{ij} \leq 1, \ \forall i \in V\},$$

Iteration 0

- *Solve the LP*

$$\bar{z}^0 = \max\{\sum_{(i,j)\in E} w_{ij}x_{ij} \mid x \in P^0\}$$

which returns an optimal solution

$$x^0 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

with value $z^0 = 15$. *(The first five coordinates correspond to the "outer cycle" of the example* $x_{12}, x_{23}, x_{34}, x_{45}, x_{15}$*, the last five coordinates to the "inner cycle"* $x_{68}, x_{8,10}, x_{10,7}, x_{79}, x_{96}$ *and and the coordinates in the middle represent the remaining edges.)*

- *The separation problem over the family* $\mathcal{F}$ *of odd set inequalities returns for example*

$$\pi^T x = x_{12} + x_{23} + x_{34} + x_{45} + x_{15} \leq 2 = \pi_0$$

which cuts off x^0 *(indeed,* $x^0_{12} + x^0_{23} + x^0_{34} + x^0_{45} + x^0_{15} = 2.5 > 2$*).*

- *Set* $P^1 = P^0 \cap \{x \mid x_{12} + x_{23} + x_{34} + x_{45} + x_{15} \leq 2\}$.

Iteration 1

- *Solve the LP*

$$\bar{z}^1 = \max\{\sum_{(i,j)\in E} w_{ij}x_{ij} \mid x \in P^1\}$$

which has optimal solution

$$x^1 = \left(1, 0, 1, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

with value $z^1 = 13.5$.

- *The separation problem over the family* $\mathcal{F}$ *of odd set inequalities returns*

$$\pi^T x = x_{68} + x_{8,10} + x_{10,7} + x_{79} + x_{96} \leq 2 = \pi_0$$

which cuts off x^1.

- *Set* $P^2 = P^1 \cap \{x \mid x_{68} + x_{8,10} + x_{10,7} + x_{79} + x_{96} \leq 2\}$.

Iteration 2

- *Solve the LP*

$$\bar{z}^2 = \max\{\sum_{(i,j)\in E} w_{ij}x_{ij} \mid x \in P^2\}$$

which has optimal solution

$$x^2 = (1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0)$$

with value $z^2 = 13$ *and is an optimal integral solution selecting* $M = \{x_{12}, x_{34}, x_{5,10}, x_{68}, x_{97}\}$ *and the algorithm terminates.*

3.3.2 A particular implementation: Gomory's fractional cutting plane algorithm

In this section we give a particular implementation of the generic cutting plane algorithm described above, which is based on the simplex algorithm.

Consider the integer program

$$\begin{aligned} \max \quad & c^T x \\ \text{s. t. } & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned} \tag{IP-Gomory}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Remember that any integer program can be brought to the form (IP-Gomory) by adding slack or surplus variables to inequality constraints and by splitting a variable into two variables representing its negative and positive parts if its domain is not originally restricted to be $x \geq 0$.

The idea is to first solve the LP-relaxation of (IP-Gomory) with the simplex algorithm. Let x^* be an optimal solution of the LP-relaxation associated with a basis B. Rewrite the vector of variables as $x = (x_B, x_N)$, where x_B are the basic variables and x_N the non-basic variables. If x^* is fractional, this necessarily means that there exists at least one basic variable taking fractional value. Then, we can generate a Chvátal-Gomory inequality using the constraint associated with this fractional basic variable. Let $A = (B, N)$ be the matrix of constraint coefficients, where B are the columns associated with the basic variables and N the columns associated with the non-basic variables. Let $c = (c_B, c_N)$ be the vector of coefficients, where c_B are the coefficients associated with the basic variables and c_N, the coefficients associated with the non-basic variables.

We can rewrite (IP-Gomory) using the notation of the Tableau form in the simplex algorithm as

$$\begin{aligned} \max \quad & c_B^T B^{-1} b && +(c_N^T - c_B^T B^{-1} N) x_N && \\ \text{s. t. } & x_B && +B^{-1} N x_N && = B^{-1} b \\ & x \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned} \tag{IP-Gomory}$$

Using the notation $\bar{z} = c_B^T B^{-1} b$, for the value of the current solution, $\bar{c}_N^T = c_N^T - c_B^T B^{-1} N$ for the reduced costs, $\bar{b} = B^{-1} b$ for the value of the basic variables and $\bar{A}_N = B^{-1} N$ for the coefficients of the non-basic variables in the constraints, and writing each constraint separately, we can rewrite the problem as

$$\begin{aligned} \max \quad & \bar{z} + \sum_{j \in N} \bar{c}_j x_j \\ \text{s. t. } & x_{B_i} + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i && i = 1, \ldots, m \\ & x \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned} \tag{IP-Gomory}$$

If the optimal solution of the LP is not integral, this means that one of the basic variables x_{B_i} is fractional, which means that for one of the rows, $\bar{b}_i$ must be fractional and non-negative.

Take the corresponding row and apply

$$x_{B_i} + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor \tag{3.9}$$

This is a Chvátal-Gomory inequality using multipliers $u_i = 1$ and $u_k = 0$ for rows $k \neq i$, therefore it is valid for (IP-Gomory).

Moreover, it cuts off the optimal LP solution x^*. Indeed, since (3.9) is valid and $x_{B_i} + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i$ is part of the constraints, if the solution x^* is not cut off, it should satisfy both constraints and therefore their sum. However, their sum is

$$\sum_{j \in N} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq (\bar{b}_i - \lfloor \bar{b}_i \rfloor)$$

which is not satisfied because by x^* because all non-basic variables x_j^* take value zero and $\bar{b}_i - \lfloor \bar{b}_i \rfloor > 0$ by definition, because $\bar{b}_i$ is fractional and non-negative.

Observation 6. *Gomory's fractional cutting plane algorithm is nowadays implemented in most solvers (despite many backs and forths in the discussion of its usefulness, see for example [10]). When trying to solve a problem using cutting planes, there is a trade-off to be taken into account. On one hand, Gomory's fractional cutting plane algorithm strengthens the formulations very much, but on the other hand it can be time consuming to generate the inequalities and also to solve large LPs with many inequalities. This is a typical trade-off that one has to consider when working with algorithms involving cutting planes, such as branch-and-cut. An appropriate balance is usually found out experimentally.*

3.4 Which valid inequalities are better?

In the previous sections of this chapter we have discussed how to define valid inequalities and how to use them in order to generate only the most useful ones for our objective function in the context of cutting plane algorithms. In this section we want to give some hints towards which valid inequalities are "better". Although there are many ways one could approach this question and several properties one can consider to define good inequalities, we focus here on two aspects, first, on the difficulty to solve the separation problem for a given family of inequalities, and secondly on whether the inequalities are necessary to the description of our feasible integer points or not.

3.4.1 Solving the separation problem

The separation problem has to be solved many times in a cutting plane algorithm (and even more times if cutting planes are embeded in a branch-and-cut algorithm). Therefore, it is of best interest to have separation problems that can be solved fast. Corresponding algorithms are called *separation algorithms*. Having fast separation algorithms becomes crucial when considering a family $\mathcal{F}$ of exponentially many inequalities. Moreover, the separation problem can sometimes be NP-hard itself, in which case heuristic approaches can be useful. Separation algorithms are a very wide and important topic in itself. In our context we only examine an example for the stable set problem.

Example 24 (**Stable Set problem: separating odd cycle inequalities**). *Consider the Stable Set problem on a graph $G = (V, E)$. In Example 14 we introduced the* odd cycle inequalities. *As a reminder, the odd cycle inequality associated to a cycle C (of odd length) is*

$$\sum_{i \in C} x_i \leq \frac{|C| - 1}{2} \tag{3.3}$$

and is valid for $STAB(G) = conv\{x \in \{0,1\}^{|V|} \mid x_i + x_j \leq 1, \ \forall (i, j) \in E\}$.

Imagine now that we have solved the LP

$$\begin{aligned} &\max \sum_{i \in V} x_i \\ &s.\ t.\ x \in ESTAB(G) \end{aligned}$$

and that we have obtained a fractional optimal solution x^*. The key question that we want to answer in this example is: How do we solve the separation problem? *That is,* how do we find an inequality (3.3) that actually cuts off point x^* or show that such an inequality does not exist?

A natural idea is to enumerate all possible odd cycles C in G and test the corresponding odd cycle inequality on x^*. If we find an odd cycle inequality that is not satisfied by x^*, then we have found our valid inequality. If for all C the odd cycle inequalities are satisfied, than we can assert that no separating inequality exists in this family.

Pure cycle enumeration is not a good idea in practice, because we potentially have many inequalities to test. However, for odd cyle inequalities, there exists a polynomial time separation algorithm which is based on the shortest path problem. Summarizing, given x^* as an input, we want to

- *either find an odd cycle C such that*

$$\sum_{i \in C} x_i^* > \frac{|C| - 1}{2},$$

- *or show that for all odd cycle C the following inequality holds*

$$\sum_{i \in C} x_i^* \leq \frac{|C| - 1}{2}$$

This defines a whole new optimization problem: "Find a graph $C = (V(C), E(C))$ with vertex set $V(C) = \{v_1, \ldots, v_{2k+1}\}$ of odd cardinality and edge set $E(C) = \{(v_i, v_{i+1}) \mid \forall i =$

$1, \ldots, 2k\} \cup \{(v_1, v_{2k+1})\}$ *and such that the value* $\sum_{i\in C} x_i^* - \frac{|C|-1}{2}$ *is maximized". Equivalently*

$$\left\{ \max_{C:\ odd\ cycle\ in\ G} \sum_{i\in C} x_i^* - \frac{|C|-1}{2} \right\} \tag{3.10}$$

Note that now the unknown of our problem is C*, while the* x_i^* *are constants. Once we have computed the optimal solution* C^{opt} *of* (3.10)*,*

- *if the optimal value* $\sum_{i\in C^{opt}} x_i^* - \frac{|C^{opt}|-1}{2}$ *is strictly positive, then the separation algorithm returns the inequality corresponding to* C^{opt}*,*

- *else, no such inequality exists (all odd cycle inequalities are satisfied by* x^**).*

To rewrite our problem as a shortest path problem, we need to rewrite our problem as a minimization problem and define weights for the edges of C*. Problem* (3.10) *can be written as*

$$\left\{ \max_{C:\ odd\ cycle\ in\ G} \sum_{i\in C} \left(x_i^* - \frac{1}{2} \right) + \frac{1}{2} \right\} = \frac{1}{2} \left\{ \max_{C:\ odd\ cycle\ in\ G} \sum_{i\in C} (2x_i^* - 1) + 1 \right\}$$

which is equivalent to

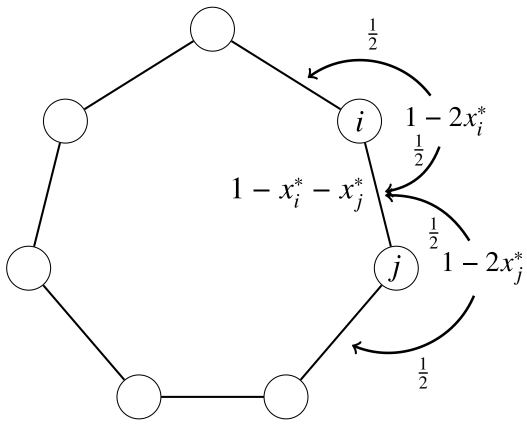
$$-\frac{1}{2} \left\{ \min_{C:\ odd\ cycle\ in\ G} \sum_{i\in C} (1 - 2x_i^*) - 1 \right\} \tag{3.11}$$

We already have the problem written on minimization form, but the objective function associates weights to vertices (to vertex i *we associate weight* $1 - 2x_i^*$*).*

For this, we rewrite the objective function based on the observation that, on a cycle C*,*

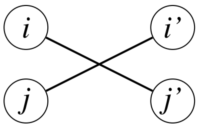
$$\sum_{i\in C} (1 - 2x_i^*) = \sum_{(i,j)\in E(C)} \left(1 - x_i^* - x_j^* \right),$$

because the weight of each vertex $i \in C$ *can be distributed half and half to both its adjacent edges in the cycle.*



In this way, we have assigned lengths $l_{ij} = 1 - x_i^* - x_j^*$ to the edges, and we want to find a shortest path with respect to l_{ij}.

The last step consists in making sure that the edges that we select for our shortest path form an odd cycle. For this, the shortest path will not directly be computed in G, but on an auxiliary graph G' constructed as follows. Let $G' = (V_1 \cup V_2, E')$ where V_1 are the original vertices, that we denote by i, and V_2 is a copy of the original vertices, that we denote by i'. If (i, j) is an edge in the original set of edges E, we construct a pair of edges (i, j') and (i', j) in G' as illustrated below.



The length of edges (i, j') and (i', j) will be l_{ij}. For each $i \in V$, compute the shortest path in G' from i to i', and select the shortest among all these paths. This can be computed in polynomial time and returns the shortest odd cycle in G, because since G' is bipartite, going from a vertex in V_1 to its copy in V_2 requires an odd number of edges.

This shortest cycle is the optimal solution of (3.11). If its length is ≥ 1, then all odd cycle inequalities are satisfied, else (if length < 1) the corresponding inequality is not satisfied and is returned by the separation algorithm.

3.4.2 Strength of valid inequalities

In this section we try to assess the quality of valid inequalities in terms of their "strength". We will formalize the idea of strength during the section.

Note first that inequalities $\pi^T x \leq \pi_0$ and $\lambda\pi^T x \leq \lambda\pi_0$ are identical for any $\lambda > 0$.

Definition 21. *Given two valid inequalities for $P \subseteq \mathbb{R}^n_{\geq 0}$, $\pi^T x \leq \pi_0$ and $\mu^T x \leq \mu_0$, we say that $\pi^T x \leq \pi_0$* dominates *$\mu^T x \leq \mu_0$ if there exists a $\lambda > 0$ such that $\pi \geq \lambda\mu$ and $\pi_0 \leq \lambda\mu_0$, and $(\pi, \pi_0) \neq (\lambda\mu, \lambda\mu_0)$.*

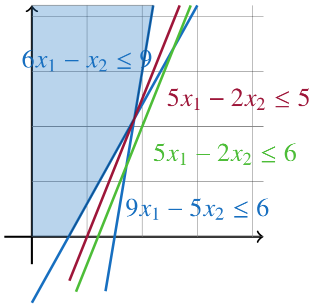
Observe that if $\pi^T x \leq \pi_0$ dominates $\mu^T x \leq \mu_0$, then $\{x \in \mathbb{R}^n_{\geq 0} \mid \pi^T x \leq \pi_0\} \subseteq \{x \in \mathbb{R}^n_{\geq 0} \mid \mu^T x \leq \mu_0\}$.

Definition 22. *A valid inequality $\pi^T x \leq \pi_0$ is* redundant *in the description of $P \subseteq \mathbb{R}^n_{\geq 0}$, if there exist $k \geq 1$ valid inequalities $\pi^{i\,T} x \leq \pi_0^i$, $i = 1, \ldots, k$ for P and weights $u_i > 0$, $i = 1, \ldots, k$ such that $\left(\sum_{i=1}^k u_i \pi^{i\,T}\right) x \leq \sum_{i=1}^k u_i \pi_0^i$ dominates $\pi^T x \leq \pi_0$.*

Observe that $\{x \in \mathbb{R}^n_{\geq 0} \mid \pi^{i\,T} x \leq \pi_0^i$, for $i = 1, \ldots, k\} \subseteq \{x \in \mathbb{R}^n_{\geq 0} \mid \left(\sum_{i=1}^k u_i \pi^{i\,T}\right) x \leq \sum_{i=1}^k u_i \pi_0^i\} \subseteq \{x \in \mathbb{R}^n_{\geq 0} \mid \mu^T x \leq \mu_0\}$. (Observe also that in the terminology of the exercises of Chapter 2 we said that $\left(\sum_{i=1}^k u_i \pi^{i\,T}\right) x \leq \sum_{i=1}^k u_i \pi_0^i$ was implied by $\pi^{i\,T} x \leq \pi_0^i$ for $i = 1, \ldots, k$.)

With these definitions, we can make our objective more precise: we would like to find descriptions of integer sets and polyhedra that do not contain redundant inequalities, or in other words, descriptions that contain only the necessary inequalities.

Example 25. *Consider $P = \{x \in \mathbb{R}^2_+ \mid 6x_1 - x_2 \leq 9,\ \ 9x_1 - 5x_2 \leq 6\}$. Inequality $5x_1 - 2x_2 \leq 6$ is valid for P, but it is redundant (take multipliers $u = \frac{1}{3}$ for both constraints defining P).*



Example 26 (Stable Set). *Consider the stable set problem on a complete graph G with 5 vertices. Inequality $x_1+x_2+x_3+2x_4+2x_5 \leq 4$ is valid for $STAB(G)$, but redundant, because it is dominated by the odd cycle inequality $x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$ and the edge inequality $x_4 + x_5 \leq 1$ (with both multipliers $u = 1$). (Update: with a rhs of 3, as in the first version, dominance condition $(\pi, \pi_0) \neq (\lambda\mu, \lambda\mu_0)$ not satisfied.)*

Remember that our objective is to find a best possible description of $conv(X)$ of an integer set X, avoiding the use of redundant inequalities. However, since $conv(X)$ is not known in general, it might be very difficult to check whether an inequality is redundant. In practice, if a formulation P for X is known, we will avoid to add inequalities to P which are dominated by already existing inequalities, which is much easier to check.

In order to better understand which inequalities are redundant and which are necessary, we introduce some definitions and results in linear algebra and polyhedral theory.

Definition 23. *A set of points $x^1, \ldots, x^k \in \mathbb{R}^n$ is* linearly independent *if the unique solution of $\sum_{i=1}^k \lambda_i x^i = 0$ is $\lambda_i = 0$, for $i = 1, \ldots, k$.*

Remember that the maximum number of linearly independent points in $\mathbb{R}^n$ is n.

Definition 24. *An* affine combination *of points $x^1, \ldots, x^k \in \mathbb{R}^n$ is a linear combination $\sum_{i=1}^k \lambda_i x^i$ with $\sum_{i=1}^k \lambda_i = 1$.*

Note that a convex combination (Definition 9) of points $x^1, \ldots, x^k \in \mathbb{R}^n$, is an affine combination with the additional requirement that $\lambda_i \geq 0$ for $i = 1, \ldots, k$.

Definition 25. *A set of points $x^1, \ldots, x^k \in \mathbb{R}^n$ is* affinely independent *if the unique solution of $\sum_{i=1}^k \lambda_i x^i = 0$ and $\sum_{i=1}^k \lambda_i = 0$ is $\lambda_i = 0$, for $i = 1, \ldots, k$.*

Remember that the maximum number of affinely independent points in $\mathbb{R}^n$ is $n + 1$.

Note that linear independence implies affine independence but the converse is not true. The following lemma is useful to check affine independence.

Lemma 1. *The following statements are equivalent*

1. $x^1, \dots, x^k \in \mathbb{R}^n$ *are affinely independent.*
2. $x^2 - x^1, \dots, x^k - x^1 \in \mathbb{R}^n$ *are linearly independent.*
3. $(x^1, 1), \dots, (x^k, 1) \in \mathbb{R}^{n+1}$ *are linearly independent.*

Definition 26. *The* dimension *of a polyhedron P, denoted by dim(P), is the maximum number of affinely independent points in P, minus one.*

By convention, we set $dim(\emptyset) = -1$. A polyhedron $P \subseteq \mathbb{R}^n$ is called *full-dimensional* if $dim(P) = n$, or equivalently, if it contains $n+1$ affinely independent points. Note that full-dimensional polyhedra have the property that there is no equation $ax = b$ satisfied at equality by all points $x \in P$. Considering full-dimensional polyhedra simplifies the notions presented here because of the following result.

Theorem 11. *If P is a full-dimensional polyhedron, then it has a unique minimal description*

$$P = \{x \in \mathbb{R}^n \mid a^i x \leq b_i \quad \text{for } i = 1, \dots, m\}$$

where each inequality is unique (except for positive multiples), and where minimal means that every inequality is necessary, that is, if for some i the corresponding inequality is removed, then the resulting set is no longer P.

Therefore, for the sake of simplicity, from now on we limit ourselves to full-dimensional polyhedra $P \subseteq \mathbb{R}^n$.

Definition 27. *A set $F \subseteq P$, $\cancel{F \neq \emptyset}$ is a* face *of P if there exists a valid inequality $\pi^T x \leq \pi_0$ for P such that $F = \{x \in P \mid \pi^T x = \pi_0\}$.*

If F is a face of P with $F = \{x \in P \mid \pi^T x = \pi_0\}$, the valid inequality $\pi^T x \leq \pi_0$ is said to represent *F.*

Note that a face F is itself a polyhedron, and therefore it has a dimension.

The face $F = \emptyset$ (of dimension -1) and the face $F = P$ (corresponding to the valid inequality $0^T x \leq 0$) are called *trivial* faces, while all other faces (with dimensions between 0 and $dim(P)-1$) are called *proper* or *non-trivial* faces.

Observation 7. *Let F_1, F_2 be non-empty faces a polyhedron $P \subseteq \mathbb{R}^n$*

$$F_1 = \{x \in P \mid \pi^1 x = \pi_0^1\}$$

$$F_2 = \{x \in P \mid \pi^2 x = \pi_0^2\}$$

If $F_1 \subset F_2$ (that is if F_1 is a proper face of the polyhedron F_2), then $dim(F_1) < dim(F_2)$.

Definition 28. *F is a* facet *of P if it is a face of P and it has dimension $dim(F) = dim(P) - 1$.*

This means that for full-dimensional polyhedra, a valid inequality $\pi^T x \leq \pi_0$ defines a facet of P if, and only if, there are n affinely independent points of P satisfying it at equality. Note also that a face F is a facet if, and only if, it is not contained in any other face $F \neq P$.

Theorem 12. *If P is full-dimensional, a valid inequality $\pi^T x \leq \pi_0$ is necessary in the description of P if, and only if, it defines a facet.*

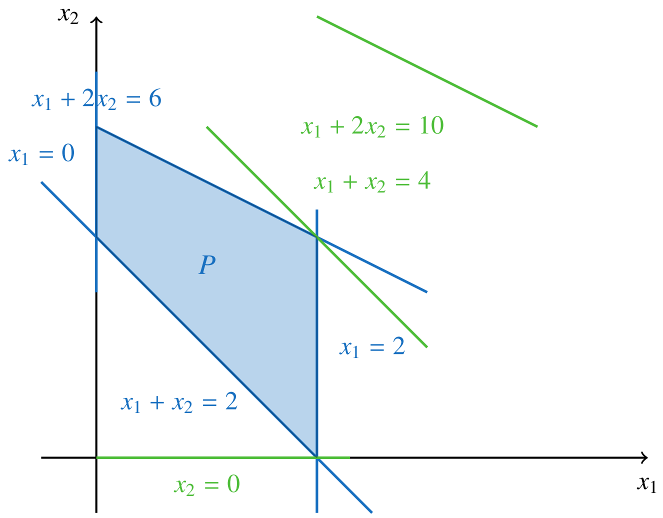
The theorem above tells us that the inequalities that describe facets are the ones that we are looking for, because they are not redundant in the description of $conv(X)$.

We will illustrate the concepts above on two examples.

Example 27. *Geometrical illustration of faces and facets. Consider the polyhedron $P \subset \mathbb{R}^2$ defined by inequalities*

$$\begin{aligned} x_1 \quad\quad\; &\leq 2 \\ x_1 + \quad x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 10 \\ x_1 + 2x_2 &\leq 6 \\ x_1 + \quad x_2 &\geq 2 \\ x_1 \quad\quad\; &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

In the illustration of P below, it is clear that only inequalities $x_1 \leq 2$, $x_1 + 2x_2 \leq 6$, $x_1 + x_2 \geq 2$, and $x_1 \geq 0$ are necessary in the description of P.



More formally, P is full-dimensional because the points $(2,0)$, $(1,1)$ and $(2,2)$ are contained in P and are affinely independent.

- *$x_1 \leq 2$ defines a facet of P because it contains $(2,0)$ and $(2,2)$, which are two affinely independent points satisfying $x_1 \leq 2$ at equality.*

- *$x_1 + 2x_2 \leq 6$ defines a facet of P because it contains $(0,3)$ and $(2,2)$, which are two affinely independent points satisfying $x_1 \leq 2$ at equality.*

- *The same reasoning can be applied to the other facet-defining inequalities.*

- *$x_1 + x_2 \leq 4$ defines a face consisting of a single point of P, point $(2, 2)$, therefore it is redundant. An alternative reasoning is that inequalities $x_1 \leq 2$ and $x_1 + 2x_2 \leq 6$ with weights $u = (\frac{1}{2}, \frac{1}{2})$ dominate $x_1 + x_2 \leq 4$.*

- *The same reasoning can be applied to the other redundant inequalities.*

Example 28. *Stable Set problem. This example will be completed at the end of the lectures. (In class exercise.)*