Proposal for a linear program to solve the 3D Bin Packing Problem

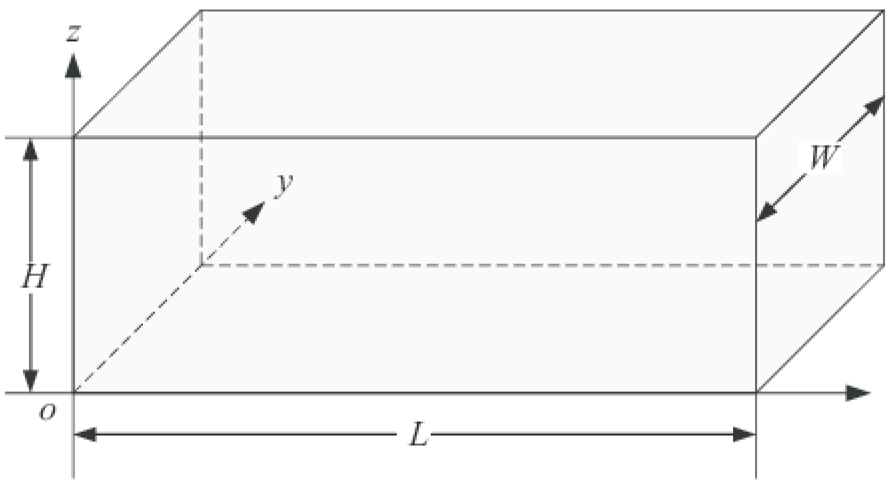
Sets:

I set of items in one order
B set of boxes
O set of orders

Idea: To work with infinite boxes of each type create $|\mathbf{I}|$ many boxes of each type. Thus we will always have enough boxes of any type.

To reduce the computational complexity we only consider one order to be packed at a time. This is possible since items of different orders are not allowed to be packed in the same box.

For positioning we will consider a normal 3D coordinate system fixed at the front-bottom-left corner of a box. Furthermore all item coordinates will be anchored on their front-bottom-left corner.
Due to the picture this linear program will consider items with a **y**-coordinate close to 0 to be in front any item with a higher **y**-coordinate.



Variables:

$\mathbf{k_b} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if box $\mathbf{b} \in \mathbf{B}$ is used
$\mathbf{v_{i,b}} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if item $\mathbf{i} \in \mathbf{I}$ is packed into box $\mathbf{b} \in \mathbf{B}$
$\mathbf{x_i} \in \mathbb{N}_{>\mathbf{0}}$**:** **x**-coordinate of item $\mathbf{i} \in \mathbf{I}$ within the box
$\mathbf{y_i} \in \mathbb{N}_{>\mathbf{0}}$**:** **y**-coordinate of item $\mathbf{i} \in \mathbf{I}$ within the box
$\mathbf{z_i} \in \mathbb{N}_{>\mathbf{0}}$**:** **z**-coordinate of item $\mathbf{i} \in \mathbf{I}$ within the box

$\mathbf{w_i^x} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if the width of the item $\mathbf{i} \in \mathbf{I}$ is aligned to the **x**-axis of the box
$\mathbf{w_i^y} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if the width of the item $\mathbf{i} \in \mathbf{I}$ is aligned to the **y**-axis of the box
$\mathbf{w_i^z} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if the width of the item $\mathbf{i} \in \mathbf{I}$ is aligned to the **z**-axis of the box

$\mathbf{h_i^x} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if the height of the item $\mathbf{i} \in \mathbf{I}$ is aligned to the $\mathbf{x}$-axis of the box
$\mathbf{h_i^y} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if the height of the item $\mathbf{i} \in \mathbf{I}$ is aligned to the $\mathbf{y}$-axis of the box
$\mathbf{h_i^z} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if the height of the item $\mathbf{i} \in \mathbf{I}$ is aligned to the $\mathbf{z}$-axis of the box

$\mathbf{l_i^x} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if the length of the item $\mathbf{i} \in \mathbf{I}$ is aligned to the $\mathbf{x}$-axis of the box
$\mathbf{l_i^y} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if the length of the item $\mathbf{i} \in \mathbf{I}$ is aligned to the $\mathbf{y}$-axis of the box
$\mathbf{l_i^z} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if the length of the item $\mathbf{i} \in \mathbf{I}$ is aligned to the $\mathbf{z}$-axis of the box

$\mathbf{a_{i,j}} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if item $\mathbf{i} \in \mathbf{I}$ is on the left of item $\mathbf{j} \in \mathbf{I}$, $\mathbf{i} \neq \mathbf{j}$
$\mathbf{b_{i,j}} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if item $\mathbf{i} \in \mathbf{I}$ is on the right of item $\mathbf{j} \in \mathbf{I}$, $\mathbf{i} \neq \mathbf{j}$
$\mathbf{c_{i,j}} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if item $\mathbf{i} \in \mathbf{I}$ is in front item $\mathbf{j} \in \mathbf{I}$, $\mathbf{i} \neq \mathbf{j}$
$\mathbf{d_{i,j}} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if item $\mathbf{i} \in \mathbf{I}$ is behind of item $\mathbf{j} \in \mathbf{I}$, $\mathbf{i} \neq \mathbf{j}$
$\mathbf{e_{i,j}} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if item $\mathbf{i} \in \mathbf{I}$ is below item $\mathbf{j} \in \mathbf{I}$, $\mathbf{i} \neq \mathbf{j}$
$\mathbf{f_{i,j}} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if item $\mathbf{i} \in \mathbf{I}$ is above item $\mathbf{j} \in \mathbf{I}$, $\mathbf{i} \neq \mathbf{j}$
$\mathbf{g_{i,j}} \in [\mathbf{0}, \mathbf{1}] = \mathbf{1}$ if item $\mathbf{i} \in \mathbf{I}$ is a different box than $\mathbf{j} \in \mathbf{I}$, $\mathbf{i} \neq \mathbf{j}$

Constants:

$\mathbf{p_b} \in \mathbb{N}_{>\mathbf{0}}$: Width of box $\mathbf{b} \in \mathbf{B}$
$\mathbf{q_b} \in \mathbb{N}_{>\mathbf{0}}$: Height of box $\mathbf{b} \in \mathbf{B}$
$\mathbf{r_b} \in \mathbb{N}_{>\mathbf{0}}$: Length of box $\mathbf{b} \in \mathbf{B}$

$\mathbf{s_i} \in \mathbb{N}_{>\mathbf{0}}$: Width of item $\mathbf{i} \in \mathbf{I}$
$\mathbf{t_i} \in \mathbb{N}_{>\mathbf{0}}$: Height of item $\mathbf{i} \in \mathbf{I}$
$\mathbf{u_i} \in \mathbb{N}_{>\mathbf{0}}$: Length of item $\mathbf{i} \in \mathbf{I}$

M: Some $\mathbf{M}$ with sufficient size. E.g. the highest dimension of the biggest box

Functions:

Objective Function:

$$\min \sum_{\mathbf{b} \in \mathbf{B}} \mathbf{k_b}$$

subject to:

All items $\mathbf{i} \in \mathbf{I}$ have to be packed into a box $\mathbf{b} \in \mathbf{B}$:

$$\sum_{\mathbf{b} \in \mathbf{B}} \mathbf{v_{i,b}} = \mathbf{1} \quad \forall \mathbf{i} \in \mathbf{I}$$

Items $\mathbf{i} \in \mathbf{I}$ may only be packed into a box $\mathbf{b} \in \mathbf{B}$ if this box is used:

$$\mathbf{v_{i,b}} \leq \mathbf{k_b} \quad \forall \mathbf{i} \in \mathbf{I}, \forall \mathbf{b} \in \mathbf{B}$$

Items in the same box are not allowed to overlap. Therefore we have to check their boarders:

$\mathbf{i} \in \mathbf{I}$ is left of $\mathbf{j} \in \mathbf{I}$:

$$\mathbf{x_i + s_i \cdot w_i^x + t_i \cdot h_i^x + u_i \cdot l_i^x \leq x_j + (1 - a_{ij}) \cdot M \quad \forall i, j \in I,\ i \neq j}$$

$\mathbf{i} \in \mathbf{I}$ is right of $\mathbf{j} \in \mathbf{I}$:

$$\mathbf{x_j + s_j \cdot w_j^x + t_j \cdot h_j^x + u_j \cdot l_j^x \leq x_i + (1 - b_{ij}) \cdot M \quad \forall i, j \in I,\ i \neq j}$$

$\mathbf{i} \in \mathbf{I}$ is in front of $\mathbf{j} \in \mathbf{I}$:

$$\mathbf{y_i + s_i \cdot w_i^y + t_i \cdot h_i^y + u_i \cdot l_i^y \leq y_j + (1 - c_{ij}) \cdot M \quad \forall i, j \in I,\ i \neq j}$$

$\mathbf{i} \in \mathbf{I}$ is behind of $\mathbf{j} \in \mathbf{I}$:

$$\mathbf{y_j + s_j \cdot w_j^y + t_j \cdot h_j^y + u_j \cdot l_j^y \leq y_i + (1 - d_{ij}) \cdot M \quad \forall i, j \in I,\ i \neq j}$$

$\mathbf{i} \in \mathbf{I}$ is below of $\mathbf{j} \in \mathbf{I}$:

$$\mathbf{z_i + s_i \cdot w_i^z + t_i \cdot h_i^z + u_i \cdot l_i^z \leq z_j + (1 - e_{ij}) \cdot M \quad \forall i, j \in I,\ i \neq j}$$

$\mathbf{i} \in \mathbf{I}$ is above of $\mathbf{j} \in \mathbf{I}$:

$$\mathbf{z_j + s_j \cdot w_j^z + t_j \cdot h_j^z + u_j \cdot l_j^z \leq z_i + (1 - f_{ij}) \cdot M \quad \forall i, j \in I,\ i \neq j}$$

Any item must have at least one positional relation to any other item in the same box:

$$\mathbf{a_{i,j} + b_{i,j} + c_{i,j} + d_{i,j} + e_{i,j} + f_{i,j} \geq 1 - g_{i,j} \quad \forall i, j \in I,\ i \neq j}$$

If two items $i, j \in I$ are not in the same box $g_{i,j}$ must be 1 and 0 otherwise:

$$\mathbf{g_{i,j}} \leq \mathbf{0.5} \cdot \left(\sum_{\mathbf{b} \in \mathbf{B}} \mathrm{abs}(\mathbf{v_{i,b} - v_{j,b}}) \right) \quad \forall \mathbf{i, j \in I,\ i \neq j}$$

No item $\mathbf{i} \in \mathbf{I}$ may be packed over the boundaries of its package:

Width:

$$\mathbf{x_i + s_i \cdot w_i^x + t_i \cdot h_i^x + u_i \cdot l_i^x \leq p_b + (1 - v_{i,b}) \cdot M \quad \forall i \in I,\ \forall b \in B}$$

Height:

$$\mathbf{y_i + s_i \cdot w_i^y + t_i \cdot h_i^y + u_i \cdot l_i^y \leq q_b + (1 - v_{i,b}) \cdot M \quad \forall i \in I,\ \forall b \in B}$$

Length:

$$\mathbf{z_i + s_i \cdot w_i^z + t_i \cdot h_i^z + u_i \cdot l_i^z \leq r_b + (1 - v_{i,b}) \cdot M \quad \forall i \in I,\ \forall b \in B}$$

Only one packing orientation per Item $i \in I$ is possible:

$$\mathbf{w_i^x + w_i^y + w_i^z = 1 \quad \forall i \in I}$$

$$\mathbf{h_i^x + h_i^y + h_i^z = 1 \quad \forall i \in I}$$

$$\mathbf{l_i^x + l_i^y + l_i^z = 1 \quad \forall i \in I}$$

$$\mathbf{w_i^x + h_i^x + l_i^x = 1 \quad \forall i \in I}$$

$$\mathbf{w_i^y + h_i^y + l_i^y = 1 \quad \forall i \in I}$$

$$\mathbf{w_i^z + h_i^z + l_i^z = 1 \quad \forall i \in I}$$

Constraints to position the items within each box were inspired by N. Z. Hu, H. L. Li and J. F. Tsai, "Solving Packing Problems by a Distributed Global Optimization Algorithm", 2012